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# ON THE EXISTENCE OF SOLUTIONS OF THE COMPOSITE TYPE THIRD ORDER EQUATION IN AN UNLIMITED DOMAIN

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Abstract. In the paper the first boundary problems for the third order equations of composite type is considered. Theorems of existence are proved in the classes of growing functions at infinity.

Keywords: PDE of composite type, existence theorems, third order PDE, generalized solution.

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## 1. Introduction

The Saint-Venant's principle (see [4], [13]) is expressed in the planar theory of elasticity as a priory estimate for a solution of a biharmonic equation satisfying homogeneous boundary conditions of the first boundary value problem in the part of the domain boundary. Such energetic estimates were obtained first in [1], [6]. These estimates do not take into account character of transformation of the body form at moving off from those parts of the bound where exterior forces are applied. In [10], another proof of the Saint-Venant's principle in the planar theory of elasticity was given. The energetic estimate obtained in this connection considered character of transformation of the body form. As a corollary of this estimate, the uniqueness theorem for the first boundary value problem of the planar theory of elasticity in unlimited domains and also Pharagmen-Lindelof type theorems were obtained. Some Pharagmen-Lindelff type theorems were proved for equations of the theory of elasticity in [15] and for elliptic equations of higher order in [2], [3], [8]. The Saint-Venant's principle for a cylindrical body was proved in [14]. An analog of the Saint-Venant's principle, uniqueness theorems in unlimited domains, and Pharagmen-Lindelff type theorems were obtained for the system of equations of the theory of elasticity in [9], [11] in the case of space with boundary conditions of the first boundary value problem. For the mixed problems similar results were derived in [12].

In the all above mentioned references method a was given for investigation in the case of even order equations. But in the case of odd order, particularly for the third order equations the methods are in the stage of development. This fact may be explained with non-symmetry of boundary conditions.

In [7] a method was given for constructing of weak solutions of boundary value problems for composite type third order equations in bounded domains. Further in the work [5] we proved uniqueness theorems in classes of functions increasing in infinity depending on the geometric characteristics of the domain.

In the present paper is presented the method of solution of first boundary value problem for the third order composite type equation. The method used in the present paper is applicable for the Korteveg - de Vriese equations with any number of variables and for the composite type equations of any odd order.

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## 2. Notations and formulation of the problem

The purpose of present paper is the investigation of the existence question of the generalized solution of the equation

$$lAu + Bu = f(x), \tag{1}$$

$$u|_{\sigma_0\bigcup\sigma_1\bigcup\sigma_2} = 0, l_0 u|_{\sigma_1} = 0, \tag{2}$$

in the unbounded domain $\Omega \subset \mathbb{R}^n_+ = \{x : x_1 > 0\}$, where

$$Au = a^{ij}(x)u_{x_i x_j} + a^i(x)u_{x_i} + a(x)u,$$

$$Bu = b^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i} + b(x)u,$$

$$lu = l_0 u + \alpha(x)u, l_0 u = \alpha^k(x)u_{x_k},$$

$$\sigma_0 = \{x \in \Gamma : \alpha^k(x)\nu_k(x) = 0\}, \sigma_1 = \{x \in \Gamma : \alpha^k(x)\nu_k(x) > 0\},$$
$$\sigma_2 = \{x \in \Gamma : \alpha^k(x)\nu_k(x) < 0\},$$

$\Gamma = \partial\Omega, \nu(x) = (\nu_1(x),\ \nu_2(x),\ ...,\ \nu_n(x))$-unit vector of the interior normal to $\Gamma$ in the point $x$.

Here and later we suppose that the summation is done on repeating indices from 1 to $n$.

Note that in the case of limited domains the existence question for the generalized solution of the problem (1), (2) is investigated in [7].

Assume that the hyper-surface $\Gamma$ is represented as $x_j = \chi(x_1,x_2, ..., x_{j-1}, x_{j+1}, ..., x_n)$, at any it's point for some $\chi$, belonging to the class $C^2$.

Suppose that all coefficients in (1) and their derivatives which occur below are bounded and measured in any finite sub-domain of the domain $\Omega$.

We will mean everywhere that

$$a^{ij} = a^{ji}, a_0 |\xi|^2 \leq a^{ij}\xi_i\xi_j \leq a_1 |\xi|^2, c^{ij} = c^{ji},$$

$$c_0 |\xi|^2 \leq c^{ij}\xi_i\xi_j \leq c_1 |\xi|^2, d_0 |\xi|^2 \leq d^{ij}\xi_i\xi_j \leq d_1 |\xi|^2,$$

$$\sum_{k=1}^{n}\left[\alpha^k(x)\right]^2 \neq 0, c^{ij} - \alpha^i a - c^i < 0, \alpha^1(x) \neq 0, q^{ij} = c - \frac{1}{2}c^i_{x_i} + \frac{1}{2}c^{ij}_{x_i x_j} + \frac{1}{2}(\alpha^i a)_{x_i} \leq -q_0 < 0,$$

at $\forall x \in \Omega \cup \Gamma$, $\forall \xi \in \mathbb{R}^n$. Here $a_0,\ a_1,\ d_0,\ d_1,\ c_{10},\ c_{11},\ c_0$ positive constants and

$$d^{ij} = c^{ij} - (\alpha^i a^{kj})_{x_k} + \alpha^i a^j + \frac{1}{2}(\alpha^k a^{ij})_{x_k}$$

$$c^{ij} = b^{ij} + \alpha a^{ij} - \alpha^k_{x_k} a^{ij},\ \ c^i = b^i + \alpha a^i - \alpha^k_{x_k} a^i,\ \ c = b + \alpha a - \alpha^k_{x_k} a.$$

Let $\{\Omega_\tau\}$ –be a family of limited subdomains of the domain $\Omega$, depending on the parameter $\tau \in \Pi = \{\tau : 0 \leq \tau \leq \tau_0\}$, $\tau_0 \leq \infty$ and $\Omega_\tau \subset \Omega_{\tau'}$, if $\tau < \tau'$. Denote $S_\tau = \partial\Omega_\tau \backslash \partial\Omega$. We will assume that $S_\tau$ is $(n-1)$ dimensional surface possessing the same smoothness as $\partial\Omega$ and its bound $\partial S_\tau \subset \partial\Omega$.

Suppose $\Gamma_\tau = \Gamma \bigcap \partial\Omega_\tau$, $\sigma_{0,\tau} = \{x \in \Gamma_\tau : \alpha^k(x)\nu_k(x) = 0\}$, $\sigma_{1,\tau} = \{x \in \Gamma_\tau : \alpha^k(x)\nu_k(x) > 0\}$, $\sigma_{2,\tau} = \{x \in \Gamma_\tau : \alpha^k(x)\nu_k(x) < 0\}$.

Determine $\sigma_{1,h,\tau} = \{x \in \sigma_{1,\tau} : \rho(x, \partial\sigma_{1,\tau}) > h\}$, $\sigma^h_{1,\tau} = \sigma_{1,\tau} \backslash \sigma_{1,h,\tau}$ for $h > 0$.

Let $E(\Omega_\tau)$ be a set of functions $\upsilon$ from the class $C^2(\overline{\Omega_\tau})$ such that $\upsilon = 0$ at $\Gamma_\tau$ and $l_0\upsilon = 0$ for some $h > 0$ at $\sigma_{0,\tau} \bigcup \sigma_{2,\tau} \bigcup \sigma^h_{1,\tau}$.

Denote by $H(\Omega_\tau)$ the completion $E(\Omega_\tau)$ on the norm

$$||u||_{H(\Omega_\tau)} = \left\{ \int\limits_{\Omega_\tau} (d^{ij} u_{x_i} u_{x_j} + u^2) dx + \int\limits_{\sigma_{1,\tau}} \alpha^k \nu_k a^{ij} u_{x_i} u_{x_j} ds \right\}^{\frac{1}{2}}.$$

We shall consider the bilinear form

$$a(u, \upsilon) = \int\limits_{\Omega_\tau} [\alpha^k a^{ij} u_{x_i} \upsilon_{x_j x_k} + (\alpha^k a^{ij})_{x_j} u_{x_i} \upsilon_{x_k} - \alpha^k a^i u_{x_i} \upsilon_{x_k} - c^{ij} u_{x_i} \upsilon_{x_j}] dx +$$

$$+ \int\limits_{\sigma_1} [(c^{ij}_{x_j} - \alpha^i a - c^i) u \upsilon_{x_i} + (c - c^i_{x_i} + c^{ij}_{x_i x_j}) u \upsilon] dx. \tag{3}$$

**Definition 2.1.** *The function $u(x)$ is called a generalized solution of the problem* (1), (2) *in the domain $\Omega$ if $u(x) \in H(\Omega_\tau)$ for any finite subdomain $\Omega_\tau$ of the domain $\Omega$ and*

$$a(u, \upsilon) = \int\limits_{\Omega_\tau} f \upsilon dx \tag{4}$$

*for an arbitrary function $\upsilon(x) \in E(\Omega_\tau)$, $\upsilon = 0$ in $S_\tau$.*

## 3. Main results

Now let $\alpha^k = const\ k = \overline{1,\ n}$, $\alpha^1 > 0$. Then it's proved in [14] the acceptance of the second condition from (2) as the generalized solution on the average.

Assume $S_\tau = \Omega \bigcap \{x : x_1 = \tau + \gamma\}$ for any $0 \le \tau \le \tau_0$, where $\gamma = const > 0$, for simplicity of the exposition.

Introduce the following notations

$$Q(u) = d^{ij} u_{x_j} u_{x_j} - q^{ij} u^2, g = a_1 d_0^{-\frac{1}{2}} (\alpha^1)^2,$$

$$P(\tau) = \sup_{S_\tau} B(x), \tag{5}$$

$$B(x) = \{2^{-1}(\alpha^1 a^1 + c^{i1} - (\alpha^1 a^{ij})_{x_j}), 0\}, \tag{6}$$

Let

$$0 < \lambda(\tau) \le \inf_{\upsilon \in N} \left\{ \int\limits_{S_\tau} Q(\upsilon) dx' \left| \int\limits_{S_\tau} \upsilon^2 dx' \right|^{-1} \right\}, x' = (x_2, ..., x_n). \tag{7}$$

Here $N$ is the set of functions $\upsilon(x)$ which are continuously differentiable on the neighborhood $S_\tau$ as $x \in \bar{\Omega}$, and are equal to 0 in $S_\tau \bigcap \Gamma$.

Let $\Phi(\tau)$ be a positive function as $\tau \in \Pi$, such that

$$\Phi(\tau) \ge g \lambda^{-\frac{1}{2}}(\tau) + P(\tau) \lambda^{-1}(\tau). \tag{8}$$

Now using the properties of this function we define two types of domains.

A) The first class of domains is expanding domains which satisfies the following condition

$$\frac{d\Phi(\tau)}{d\tau} \ge \varepsilon, \forall \tau \in \Pi, \varepsilon = const, 0 < \varepsilon < 1;$$

These domains are sited out of some cone at infinity. Here $\tau(\beta)$ is a solution of the equation

$$\frac{d\tau}{d\beta} = \frac{\Phi}{\varepsilon\tau + \Phi_\tau} \tag{9}$$

with initial condition $\tau(0) = 0$.

B) The second class of domains is domains which satisfies the following condition

$$\frac{d\Phi(\tau)}{d\tau} \leq \varepsilon, \forall\tau \in \Pi, \varepsilon = const, 0 < \varepsilon < 1;$$

These are domains inclined in some cone. Here $\tau(\beta)$ is a solution of the equation

$$\frac{d\tau}{d\beta} = \frac{\Phi}{\varepsilon\tau + \varepsilon} \tag{10}$$

with initial condition $\tau(0) = 0$.

In both cases the function $\Phi(\tau)$ is such that the right parts of (9) and (10) will be absolutely continuous.

Next theorem is proved in [5].

**Theorem 3.1.** *(The analogue of the Saint-Venant's principle). Let $u(x)$ be a generalized solution of the problem (1), (2) from the class A) in the domain $\Omega$, moreover*

$$(\alpha^1 a^{ij})_{x_i} - (\alpha^1 a^{ij})_{x_i x_j} + 3c^{ij}_{x_i x_j} - 2c^1 - 2\alpha^i a \geq 0 \, and \, f(x) = 0 \, in \, \Omega_{\tau_0}. \tag{11}$$

*Then the following estimation*

$$\int\limits_{\Omega_{\tau(R_0)}} Q(u)dx \leq \exp\left\{-\varepsilon \int\limits_{\tau(R_0)}^{\tau(R)} \frac{s}{\Phi(s)} ds\right\} \int\limits_{\Omega_{\tau(R)}} Q(u)dx \tag{12}$$

*is valid for any $R_0$ and $R$ such that $0 \leq R_0 \leq R$.*

Now we will prove theorems of existence for the problem (1), (2) in the unbounded domains.

**Lemma 3.1.** *Suppose that there exists an unlimited sequence of finite subdomains $\Omega_N$ of the domain $\Omega$ from the class A) such that $\Omega_N \subset \Omega_{N+1}$ as $N = 1, 2, ...$; $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$. Let the estimation*

$$\int\limits_{\Omega_N} Q(\omega)dx \leq \exp\left\{-\varepsilon \int\limits_{N}^{N+1} \frac{sds}{\Phi(s)}\right\} \int\limits_{\Omega_{N+1}} Q(\omega)dx. \tag{13}$$

*holds for every fixed $N$ and for any function $\omega$, which is the generalized solution of the equation (1) in $\Omega_{N+1}$, as $f \equiv 0$, with boundary conditions (2) at $\bar{\Omega}_{N+1} \cap \Gamma$. Let the function $f(x)$ be determined in $\Omega$ and its growth satisfies to the following condition*

$$\int\limits_{\Omega_N} f^2 dx \leq M_1 \Lambda(\Omega_N) \exp\left\{(1-\delta) \int\limits_{0}^{N} \frac{\varepsilon s}{\Phi(s)} ds\right\}, \quad N = 1, 2, ..., \tag{14}$$

*where $\delta = const$, $0 < \delta < 1$, the constant $M_1$ doesn't depend on $N$,*

$$\Lambda(\Omega_N) = \inf_{v \in H(\Omega_N)} \left\{ \int\limits_{\Omega_N} Q(v)dx \left| \int\limits_{\Omega_N} v^2 dx \right|^{-1} \right\}. \tag{15}$$

*Then there exists the unique generalized solution $u(x)$ of the problem (1), (2) and the estimation*

$$\int\limits_{\Omega_N} Q(\omega)dx \leq M_2 \exp\left\{(1-\delta)\int\limits_0^N \frac{\varepsilon s}{\Phi(s)}ds\right\}, \quad N = 1,\ 2,\ ..., \tag{16}$$

*is valid for $u(x)$, where the constant $M_2$ doesn't depend on $N$.*

*Proof.* Note that $\Lambda(\Omega_N) > 0$. Denote by $u_l^m(x)$ a sequence of functions from the class $E(\Omega_l)$ such that $u_l^m(x)$ converges to $u_l(x)$ on the norm $H(\Omega_l)$, as $m \to \infty$ and $u_l(x)$ is the generalized solution of the problem (1), (2) in the domain $\Omega_l$ (see [14]). Fix an arbitrary subdomain $\Omega_N$ from the sequence $\Omega_1 \subset \Omega_2 \subset ...$ and consider the sequence of the subdomains $\Omega_{N+k}$, $k \to \infty$.

We have

$$\int\limits_{\Omega_{N+k}} Q(u_{N+k})dx = -\int\limits_{\Omega_{N+k}} f u_{N+k} dx.$$

Integrating (4) by parts for $\Omega_\tau = \Omega_{N+k}$, $u = u_{N+k}$, $v = u_{N+k}^m$, and taking the limit as $m \to \infty$.

Hence using the Cauchy-Bunyakowsky inequality we have

$$\int\limits_{\Omega_{N+k}} Q(u_{N+k})dx \leq \left(\int\limits_{\Omega_{N+k}} f^2 dx\right)^{\frac{1}{2}} \left(\int\limits_{\Omega_{N+k}} u_{N+k}^2 dx\right)^{\frac{1}{2}},$$

$$\int\limits_{\Omega_{N+k}} Q(u_{N+k})dx \leq \Lambda^{-1}(\Omega_{N+k}) \int\limits_{\Omega_{N+k}} f^2 dx.$$

Taking into account (15) we obtain from the last inequality

$$\int\limits_{\Omega_{N+k}} E(u_{N+k})dx \leq M_1^* \exp\left\{(1-\delta)\int\limits_0^{N+k} \frac{\varepsilon s}{\Phi(s)}ds\right\}. \tag{17}$$

Set

$$\langle v \rangle_{\Omega_l} = \left(\int\limits_{\Omega_l} Q(v)dx\right)^{\frac{1}{2}}.$$

It is easy to see that the function $\omega = u_{N+k+1} - u_{N+k}$ is the generalized solution of the equation (1) satisfying to the boundary condition (2) in the domain $\Omega_{N+k}$ as $f(x) = 0$. So applying estimation (14) to domains $\Omega_N$, $\Omega_{N+1}$, ..., $\Omega_{N+k}$, consecutively and taking into account the inequality (18), we will find out that

$$\langle u_{N+k+1} - u_{N+k} \rangle_{\Omega_N} \leq M_3 \exp\left\{-\frac{\delta}{2}\int\limits_N^{N+k} \frac{\varepsilon s}{\Phi(s)}ds\right\}, \tag{18}$$

where the constant $M_3$ doesn't depend from $N$ and $k$.

It follows from the inequality (18)

$$\langle u_{N+p+q} - u_{N+p} \rangle_{\Omega_N} \leq M_4 \exp\left\{-\frac{\delta}{2}\int\limits_N^{N+p} \frac{\varepsilon s}{\Phi(s)}ds\right\}, \tag{19}$$

for every integer numbers $p > 0$, $q > 0$ , where the constant $M_4$ doesn't depend from $N$, $p$ and $q$. Hence $\langle u_{N+p+q} - u_{N+p}\rangle_{\Omega_N} \to 0$ for any $q > 0$, $p \to \infty$.

For any bounded subset $\Omega_\tau$ of the domain $\Omega$ and for arbitrary set $\mu \subset \partial\Omega'$ we denote by $H^1(\Omega_\tau,\ \mu)$ completion of $E(\Omega_\tau,\ \mu)$ with respect to norm

$$||\omega||_{H^1(\Omega_\tau,\ \mu)} = \left\{ \int\limits_{\Omega_\tau} (d^{ij}\omega_{x_i}\omega_{x_j} + \omega^2)dx \right\}^{\frac{1}{2}}.$$

Since $H^1(\Omega_\tau,\ \mu)$ is a Hilbert space and

$$\|u_{N+p} - u_{N+p+q}\|^2_{H^1(\Omega_N)} \le M_5 \langle u_{N+p} - u_{N+p+q}\rangle^2_{\Omega_N},$$

where the constant $M_5$ doesn't depend on $p$, then the sequence $\{u_p\}$ converges to the function $u(x) \in H^1(\Omega_N,\ \mu)$ as $p \to \infty$ in the norm $H^1(\Omega_\tau,\ \mu)$.

Since $N > 0$ is chosen arbitrary then $u(x)$ is defined in $\Omega$, and $u(x) \in H^1(\Omega')$, for any bounded subdomain $\Omega'$ of the domain $\Omega$.

By virtue of known theorems of inclusion the trace of the function $u(x)$ exists on $\partial\Omega' \cap \Gamma$ and the first condition from (2) holds in this set. The acceptance the second condition from (2) as a generalized solution is proved in [14].

It is easy to see that $u_{N+p_m}$ satisfies to the integral identity (4) for $\Omega_\tau = \Omega_N$ and any $\upsilon \in E(\Omega_N)$. Taking the limit as $p_m \to \infty$, we'll get $u(x)$ satisfies the integral identity (4) for $\Omega_\tau = \Omega_N$, and $\upsilon \in E(\Omega_N)$. Since $N > 0$ is arbitrary then it follows that the integral identity (4) is valid for any limited subdomain $\Omega_N \subset \Omega$ and arbitrary function $\upsilon \in E(\Omega_N)$. So $u(x)$ is a generalized solution of the problem (1), (2) in the domain $\Omega$.

Take $p = 1$ in (20). Let $q$ tends to $\infty$. Using the estimation (18) we will find out

$$\langle u\rangle_{\Omega_N} \le k \exp\left\{ \frac{1-\delta}{2} \int\limits_0^N \frac{\varepsilon s}{\Phi(s)} ds \right\} + \langle u_{N+1}\rangle_{\Omega_N} \le$$

$$\le k \exp\left\{ \frac{1-\delta}{2} \int\limits_0^N \frac{\varepsilon s}{\Phi(s)} ds \right\} + k_1 \exp\left\{ \frac{1-\delta}{2} \int\limits_0^{N+1} \frac{\varepsilon s}{\Phi(s)} ds \right\}.$$

Now we will show that the solution $u(x)$ of the problem (1), (2) satisfying to (17) is unique. It follows from (14) that $u(x)$ and $\upsilon$ are the generalized solutions of the problem (1), (2) for which the estimation (17) holds in the domain $\Omega$, then for any $l > 0$

$$\int\limits_{\Omega_l} Q(u-\upsilon)dx \le \exp\left\{ -\int\limits_l^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} \int\limits_{\Omega_{l+j}} Q(u-\upsilon)dx \le$$

$$\le \exp\left\{ -\int\limits_l^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} \left[ \int\limits_{\Omega_{l+j}} E(u)dx + \int\limits_{\Omega_{l+j}} E(\upsilon)dx \right] \le$$

$$\le \exp\left\{ -\int\limits_l^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} \left[ M_5 \exp\left\{ (1-\delta) \int\limits_0^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} + M_5^* \exp\left\{ (1-\delta) \int\limits_0^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} \right] \le$$

$$\le 2M_5' \exp\left\{ -\int\limits_l^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} \exp\left\{ (1-\delta) \int\limits_0^{l+j} \frac{\varepsilon s}{\Phi(s)} ds \right\} \le$$

$$\leq M_6 \exp\left\{-\delta \int\limits_{l}^{l+j} \frac{\varepsilon s}{\Phi(s)} ds\right\},$$

where the constant $M_6 = \max\{M_5, M_5^*\}$ doesn't depend from $j$. Taking the limit in the last inequality as $j \to \infty$, we will obtain that $\langle u - v\rangle_{\Omega_l} = 0$ for any $l > 0$. Since $\Lambda(\Omega_l) > 0$, then $u - v \equiv 0$ in $\Omega_l$ and so $u = v$ in $\Omega_l$.

One can see from the lemma that the existence of generalized solution of the problem (1), (2) in $\Omega$ is based on the assumption that the relations (14) are realized.

Now we will show how one can construct the sequence of the domains $\{\Omega_N\}$ for which these relations are valid. Let $\{\Omega_\tau\}$ be a family of limited subdomains of the domain $\Omega$ and let all conditions of theorem 3.1 hold. Then the estimation

$$\int\limits_{\Omega_{\tau(R_0)}} Q(u)dx \leq \exp\left\{-\int\limits_{\tau(R_0)}^{\tau(R)} \frac{\varepsilon s}{\Phi(s)} ds\right\} \int\limits_{\Omega_{\tau(R)}} Q(u)dx \tag{20}$$

takes place for any $R_0$ and $R$ such that $0 \leq R_0 \leq R < R^*$. □

The last inequality implies the rule of construction of the domains $\Omega_N$.

The following theorem of existence and uniqueness follows from Lemma 1 and the estimation (21).

**Theorem 3.2.** *Let $\Lambda\left(\Omega_{\tau(k)}\right) > 0$ for every domain $\Omega_{\tau(k)}$ and let the function $f(x)$ is defined in $\Omega$, satisfies the relations*

$$\Lambda^{-1}\left(\Omega_{\tau(k)}\right) \int\limits_{\Omega_{\tau(k)}} f^2 dx \leq M_7 \exp\left\{(1-\delta) \int\limits_{0}^{\tau(k)} \frac{\varepsilon s}{\Phi(s)} ds\right\}, k = 1, 2, ...,$$

*where $\delta = const$, $0 < \delta < 1$, the constant $M_7$ doesn't depend from $k$, $\tau(k)$ is the function defined in* (13). *Then there exists the unique generalized solution $u(x)$ of the problem* (1), (2), *for which inequalities*

$$\int\limits_{\Omega_{\tau(k)}} Q(u)dx \leq M_8 \exp\left\{(1-\delta) \int\limits_{0}^{\tau(k)} \frac{\varepsilon s}{\Phi(s)} ds\right\}, k = 1, 2, ....$$

*Proof.* Let $\Omega_j = \Omega_{\tau(j)}$ in Lemma 1. Inequalities (14) follows from Theorem 3.1. Thus the statement of the Theorem 3.2 follows from Lemma 1. □

The following theorem is valid for the domains from class B).

**Theorem 3.3.** *(The analogue of the Saint-Venant's principle). Let $u(x)$ be a generalized solution of the problem* (1), (2) *from the class B) in the domain $\Omega$, moreover condition* (11) *holds. Then the following estimation*

$$\int\limits_{\Omega_{\tau(R_0)}} Q(u)dx \leq \frac{\tau(R)+1}{\tau(R_0)+1} \exp[-(R-R_0)] \int\limits_{\Omega_{\tau(R)}} Q(u)dx. \tag{21}$$

*are valid.*

**Theorem 3.4.** *Let $\Lambda\left(\Omega_{\tau(k)}\right) > 0$ for every domain $\Omega_{\tau(k)}$ and let the function $f(x)$ be defined in $\Omega$, satisfies the relations*

$$\Lambda^{-1}\left(\Omega_{\tau(k)}\right) \int\limits_{\Omega_{\tau(k)}} f^2 dx \le M_9 \exp\left\{(1-\delta)k\right\}, k = 1, 2, ...,$$

*where $\delta = const$, $0 < \delta < 1$, the constant $M_9$ doesn't depend from $k$, $\tau(k)$. Then there exists the unique generalized solution $u(x)$ of the problem (1), (2), for which inequality*

$$\int\limits_{\Omega_{\tau(k)}} Q(u)dx \le M_{10} \exp(-(1-\delta)k). \tag{22}$$

*is valid.*

Proof of Theorem 3.4 is analogous to the proof of the Theorem 3.3.

Let $\alpha^k(x)\nu_k(x) \le 0$ at $\Gamma$. Then the problem (1), (2) is turned into the Direchlet problem, in this case the operator $l_0$ has variable coefficients and the solution of the problem (1), (2) $u(x) \in \tilde{W}_2^2(\Omega_\tau)$ (see [14]). In just the same way as in the item I one can obtain the analogue of the Saint-Venant's principle for the solution and existence theorems in classes of function growing in infinity which are proved similarly to the theorems 3.1, 3.2, 3.3 and 3.4.

**Remark 3.1.** *Analogous results can be obtained when $S_\tau = \Omega \cap \{|x| = \tau + \gamma\}$.*

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